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MALNORMALITY IS UNDECIDABLE IN HYPERBOLIC GROUPS

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ABSTRACT

In answer to a question of Myasnikov, we show that there exist hyperbolic groups for which there is no algorithm to decide which finitely generated subgroups are malnormal or quasiconvex.

In the most recently published list of problems in combinatorial group theory [2], Alexei Myasnikov asked if every hyperbolic group admits an algorithm that decides which finitely generated subgroups are malnormal (problem H14). The purpose of the present note is to explain why such an algorithm does not exist in general. Our construction leaves open the question of whether there exists such an algorithm that decides which finitely presented subgroups of a hyperbolic group are malnormal.

Recall that a subgroup H of a group Γ is malnormal if for every $g \in \Gamma \setminus H$ one has $\gamma^{-1}H\gamma \cap H = \{1\}$. Malnormality is closely related to quasi-convexity and the notion of finite height and width (see [5]).

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THEOREM 1: *There exists a torsion-free hyperbolic group Γ for which there is no algorithm to decide any of the following questions for finitely generated subgroups $H \subset \Gamma$:*

(1) *Is H malnormal?*
(2) *Is H quasiconvex?*
(3) *Does H have finite width (or height)?*
(4) *Is H finitely presented?*

Moreover, one can arrange for Γ to be the fundamental group of a compact 2-dimensional complex that is negatively curved in the sense of A. D. Alexandrov.

This result is a natural addendum to [3]; indeed the essential features of our proof are to be found in that article. The idea is to apply the Rips construction in order to exploit the fact that there are finitely presented groups in which the generation problem is undecidable. For the sake of definiteness, we shall work with the following example of this phenomenon (cf. [8] page 38, [7] page 194 and [9]).

LEMMA 2: *If F is a free group of rank $n \geq 2$ and X is a finite generating set for $F \times F$, then there does not exist an algorithm to determine which finite sets S of words in the letters $X^{\pm 1}$ generate $F \times F$. Indeed there exists a recursive sequence (S_n) of finite sets of words such that one cannot determine whether $\langle S_n \rangle = F \times F$, and if equality fails then $\langle S_n \rangle$ is not finitely presented.*

Proof: It is enough to consider the case $X = \{(a,1),(1,a) \mid a \in A\}$ where A is a basis for F. Associated to each finite n-generator group presentation $\mathcal{P} = \langle A \mid R \rangle$ one has the fibre product $D = gp\{(a,a),(r,1) \mid a \in A, r \in R\} \subset F \times F$, which may be specified by writing the generators as a set of words $S(\mathcal{P}) \subset (X^{\pm 1})^*$ in the obvious manner. If the group presented by $\mathcal{P}$ is trivial then $D = F \times F$. But if the group is infinite then D is not finitely presented (see [6]).

It is well-known that there exist recursive sequences of n-generator presentations $(\mathcal{P}_n)$ such that one knows that each of the groups presented is either trivial or infinite, but there is no algorithm to recognise which are trivial. Define $S_n = S(\mathcal{P}_n)$. ∎

The Rips construction [10] is an algorithm that assigns to a finite group presentation $\mathcal{P} = \langle X \mid R \rangle$ a short exact sequence $1 \to N \to \Gamma \to P \to 1$, where P is the group presented by $\mathcal{P}$, the group N is finitely generated, by A say, $\Gamma = \langle X, A \mid \tilde{R} \rangle$ is a torsion-free hyperbolic group, and $\Gamma \to P$ maps the image in Γ of each generator $x \in X$ to its image in $P = |\mathcal{P}|$. One may modify this construction so as to arrange that Γ be the fundamental group of a compact locally CAT(-1) piecewise-hyperbolic 2-complex (see [11] or [4] page 224). One can also arrange for Γ to be a residually finite torsion-free word-hyperbolic group (see [12]).

Proof of Theorem 1: Let Γ be the group obtained by applying the Rips construction to a finite presentation of $F \times F$, where F is as in the lemma. Let S be a finite set of words in the letters $\{x, x^{-1} \mid x \in X\}$, let S^Γ be the subgroup of Γ generated by S, and let S^P be its image in $P = F \times F$. Note that NS^Γ, which is generated by the finite set $S \cup A$, is equal to Γ if and only if $S^P = P$, and if $S^P = NS^\Gamma/N$ is not finitely presented then neither is NS^Γ.

Consider the sequence (S_n) provided by the lemma. Setting $H = NS_n^\Gamma$ proves part (4) of the theorem. And since quasi-convex subgroups are finitely presented, part (2) is also proved.

It follows immediately from the definitions that if a subgroup H has infinite index in a group G, and H contains an infinite subgroup N that is normal in G, then H is not malnormal in G, indeed it does not have finite height (hence width). Thus we see that each of our sequence of subgroups $H = NS_n^\Gamma$ satisfies the conditions (1) to (4) (separately) if and only if $H = \Gamma$, which is an undecidable equality. ∎

As Alexei Myasnikov pointed out to us, by taking the free product of the group constructed in the above proof with a non-trivial hyperbolic group (for example $\mathbb{Z}$) one obtains a hyperbolic group where there is no algorithm to decide which finitely generated *proper* subgroups have the properties listed in items (1) to (4) of the above theorem. On the other hand, the following question remains unresolved:

QUESTION 3: *Does every hyperbolic group admit an algorithm that decides which of its finitely presented subgroups are malnormal?*

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